Budget Constraints

Economics 100A

Fall 2021

1 Overview

1.1 Constraints

Economics 100A considers the "demand side" of the market. By this, we mean to say that we will be discussing the consumer and their choices. Eventually, we will be deriving demand curves using a combination of utility functions and budget constraints. So what exactly is a budget constraint?

A budget constraint represents how many goods a person can afford. When we model economic agents, we often think of these agents as maximizing their happiness (i.e., their utility) subject to some constraint. After all, even if having really expensive sushi for dinner every night would give me more utility than, for example, a sandwich, it is just not feasible for me to spend that much on sushi. This leaves me in a situation where I have to optimize my income to yield maximum happiness.

Now, in reality, there are potentially hundreds of lunch items from which I may choose. Rather than calculating optimal consumption between all of these goods, we will limit our analysis to just two. Suppose we have two goods, x_1 and x_2. Both of these goods have a respective price: p_1 and p_2. These prices are for each unit of the good, so two units of x_1 would cost $2p_1$. Importantly, we also have a certain amount of money to spend on these goods. In this class, we use I to represent income. So, we have x_1, x_2, p_1, p_2, and I.

If you recall from the math review notes, when we do constrained optimization, there are four considerations:

1. **Objective Function**: What are we maximizing/minimizing?
2. **Constraint**: What do we constrain our objective function to?
3. **Choice variables**: What variables do we choose to achieve maximization/minimization?
4. **Exogenous variables**: What variables affect the situation but are taken as given?

These notes are more concerned with the constraint, choice variables, and exogenous variables. As you might have guessed, our utility functions will be our objective functions. However, we will solve those later.

Constraint

In the constraint set up, we have our budget I and two goods: x_1 and x_2. These goods both have prices. For bundle (q_1, q_2) to be feasible, it must be the case that $p_1x_1 + p_2x_2 \leq I$. Now, this assumes that the consumer cannot borrow using credit, but we will cover those more complicated models in the future. For now, we have two goods and one period. To make our lives easier, we typically equate income and total expenditure.

Here is a question to test your understanding of expenditures and income. What if we had more than two goods? Say, we have N goods. What does our constraint look like?

$$p_1x_1 + ...p_Nx_N = \Sigma p_x x_i \leq I \tag{1}$$

Choice variables
What are the variables we can choose? Well, let's think about it. Can we choose the prices of the goods? Not really, considering we do not own the goods (yet!). It's tough to say that we can choose our income, though in theory we can allocate certain portions of our income to, say, lunch. The main thing we can control is how many of goods x_1 and x_2 we purchase.

Exogenous variables
Like previously stated, we cannot really choose our income or the prices. This makes some sense, as if we did have any say in these, we would make our incomes huge and reduce prices down to \$0. Well, at least I would.

We want to remain simple in this class. Although we could allocate only part of our income to purchasing goods, we will assume in this class that a consumer allocates all of their income to goods x_1 and x_2, and that the consumer does not get to choose the prices. In other words, we take these as given in the models.

2 Drawing Budget Constraints

2.1 Overview

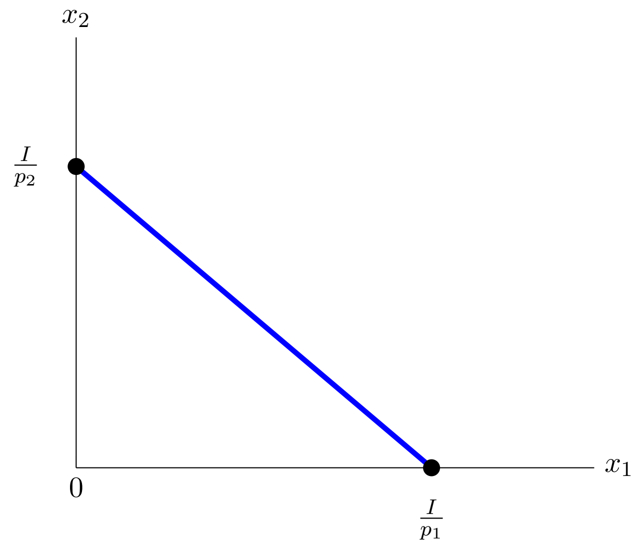
Recall that the equation for a budget constraint is

$$p_1x_1 + p_2x_2 = I \tag{2}$$

Now, how can we plot this equation in two dimensions? You should be comfortable with plotting linear functions, so I will assume you are familiar with the formula $y = mx + b$. We can actually rearrange the equation to write it in terms of x_2, since x_2 will be on the vertical (y) axis.

$$p_2x_2 = -p_1x_1I$$
$$x_2 = -\frac{p_1x_1I}{p_2}$$
$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{I}{p_2}$$

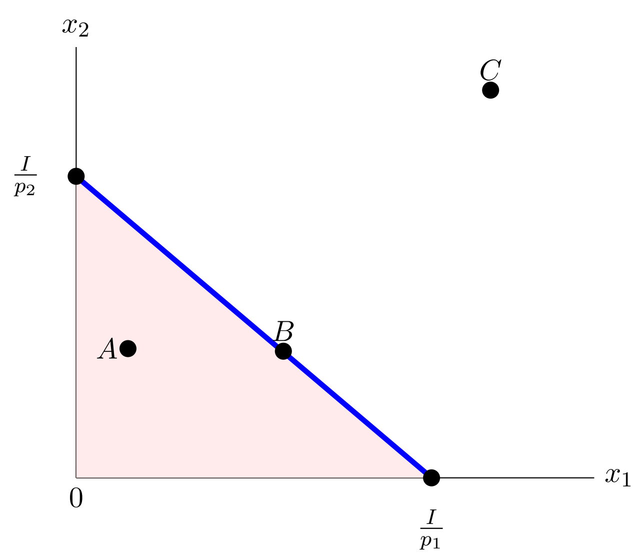
Now we see that if we want to graph a budget constraint, we first find the intercept at $\frac{I}{p_2}$ and draw the line according to the slope $-\frac{p_1}{p_2}$. For your own edification, you should prove that the x-intercept is just $\frac{I}{p_1}$.



In a relevant sense, the budget constraint limits the different bundles we can afford. In the diagram below, we have three points: A, B, and C. Notice how bundle A is below the constraint. This means that we can afford it, but it would not be optimal, as we are not spending all of our income. Anything shaded in red is affordable. However, in order to spend all of our income, we need to be consuming on the budget constraint. Thankfully, point B represents a consumer who spends all of their money. This consumer spends exactly all of their income. Finally, at point C, the consumer is spending more than their income, which in this class will pretty much be impossible.

To summarize:

1. **At point A**: The consumer is spending less than their income. $p_1x_1 + p_2x_2 < I$.
2. **At point B**: The consumer is spending exactly their income. $p_1x_1 + p_2x_2 = I$.
3. **At point A**: The consumer is spending more than their income. $p_1x_1 + p_2x_2 > I$.



Another more intuitive way of thinking about our values is that the intercept represents the maximum number of each good we can purchase. So, then, if we allocate all our our income I to a single good, it necessarily follows that we can only purchase $\frac{I}{p_x}$. If we are somewhere between the two extreme values, then we have some mix of goods that is purchased at either price. The slope here represents the change in y for a one-unit change in x. We tend to think about this as opportunity cost, as if we consume more of good x_1, we consume less of good x_2. You are giving up spending p_2 on x_2. Therefore, the slope is *necessarily* negative – since x_1 and x_2 are the only goods on which we can spend our money, it must be the case that increasing our purchases of x_1 means we spend less on x_2.

As you can probably guess, any point beyond (to the right of) the budget constraint is not feasible for the consumer. Any point below (to the left of) the budget constraint represents the consumer spending less than their income I. Any point on the line represents the consumer spending their entire income.

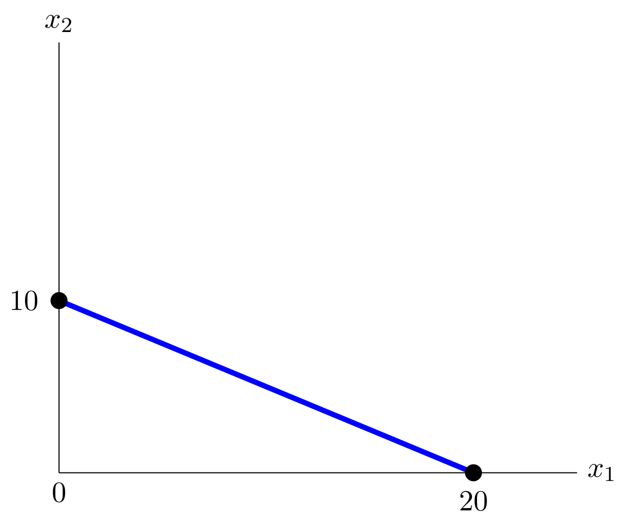
To summarize:

- **Any point to the right of the budget constraint** represents the consumer spending *more* than their income $p_1x_1 + p_2x_2 > I$
- **Any point to the left of the budget constraint** represents the consumer spending *less* than their income $p_1x_1 + p_2x_2 < I$
- **Any point to the on the budget constraint** represents the consumer spending *exactly all* their income $p_1x_1 + p_2x_2 = I$

2.2 Simple Case

In a sense, the budget constraint is the simplest graph in 100A because it is typically just a straight line. Each intercept will describe the maximum number of each good the consumer can purchase. This should make sense intuitively, as when $x_1 = 0$, x_2 is maximized and vice-versa. Let's do some simple examples.

Example one: A consumer chooses between apples (x_1) and oranges (x_2). p_1 is \$5 and p_2 is \$10. Income equals \$100. Draw the budget constraint and label the relevant points.



We can write out our constraint as follows:

$$100 = 5x_1 + 10x_2$$

Rearranging to solve for x_2:

$$x_2 = 10 - .5x_1$$

So now we see that our y-intercept will be at 10, and our constraint will have a negative slope of $-.5$. Alternatively, we could have just divided:

$$x_2 = \frac{I}{p_2} = \frac{100}{10} = 10$$

$$x_1 = \frac{I}{p_1} = \frac{100}{5} = 20$$

$$\frac{p_1}{p_2} = \frac{5}{10} = .5$$

Example two: A consumer chooses between coffee (x_1) and tea (x_2). p_1 is \$2 and p_2 is \$4. Income equals \$20. Draw the budget constraint and label the relevant points.

We can write out our constraint as follows:

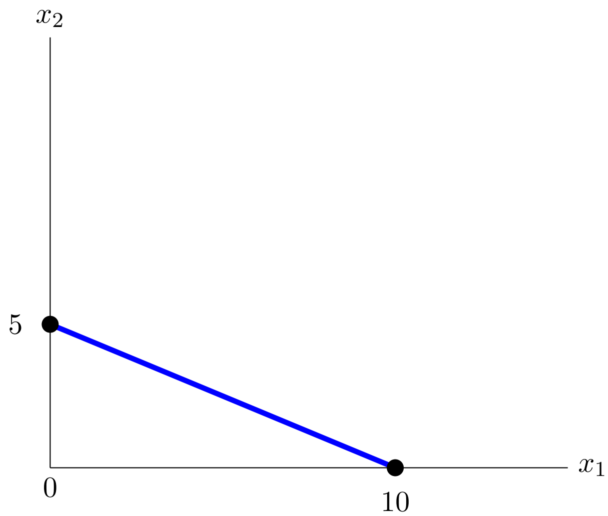
$$20 = 2x_1 + 4x_2$$

$$x_1 = \frac{20}{2} = 10$$

$$x_2 = \frac{20}{4} = 5$$

$$\frac{p_1}{p_2} = \frac{2}{4} = -2$$

Et voila. We have all we need to reasonably plot this thing!



Example three: A consumer can consume two bundles (q_1, q_2): $(8, 28)$ and $(12, 8)$. Draw the budget constraint and label the relevant points.

This is a fun, kind of cute one. We can find the slope between two points using the point-slope formula. Let's claim that $(8, 28) = (x_0, y_0)$. This means that $(12, 8) = (x_1, x_2)$.

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$m = \frac{8 - 28}{12 - 8}$$

$$m = \frac{-20}{4}$$

$$m = -5$$

So we have a slope of -5. Now what do we need? The intercept! Since we know that $y = mx + b$, we can plug in one of the points and find the y-intercept.

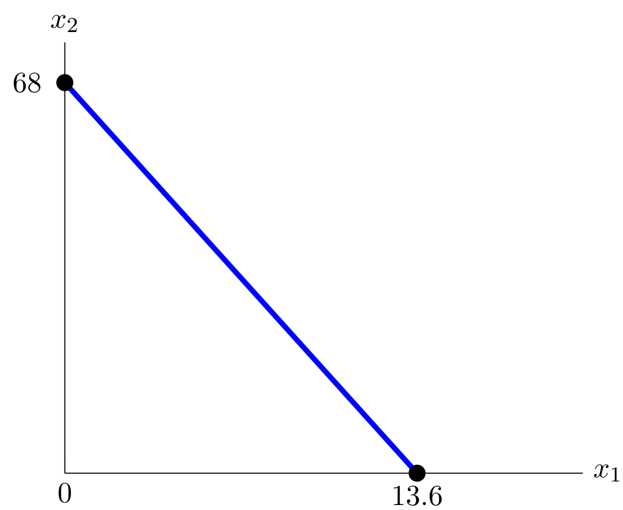
$$y = mx + b$$

$$8 = -5(12) + b$$

$$8 = -60 + b$$

$$b = 68$$

Wonderful! We found our y-intercept. We can now solve for the intercept, knowing that $y = -5x + 68$. We should get an x value of 13.6.



3 Change in Parameters

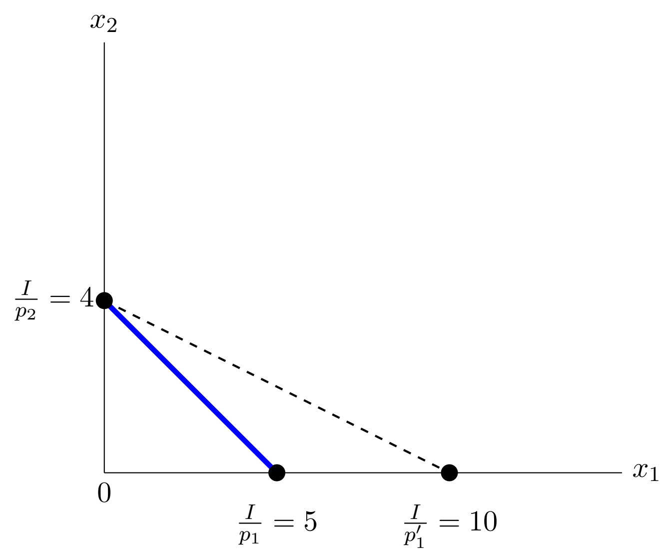
I think it is reasonable to imagine scenarios in which either prices or income change. How does the budget constraint shift, then? We can go over some specific examples first, but let's also think about how changes to our parameters change the math behind the constraint.

A change to income shifts the budget constraint out in a parallel fashion. This makes intuitive sense, because a consumer is either richer or poorer, but the slope of the line has not changed at all. In other words, if prices remain constant, the opportunity cost is the same for both goods.

A change in prices causes the budget constraint to pivot inward or outward. Convince yourself of this using a basic example. If the price of a good doubles, you can afford fewer of that good, meaning the intercept should shift inward/toward the origin. This represents a shift in the slope as well, since a slope parameter changes. Economically, think about this as the opportunity cost of consuming either good changing.

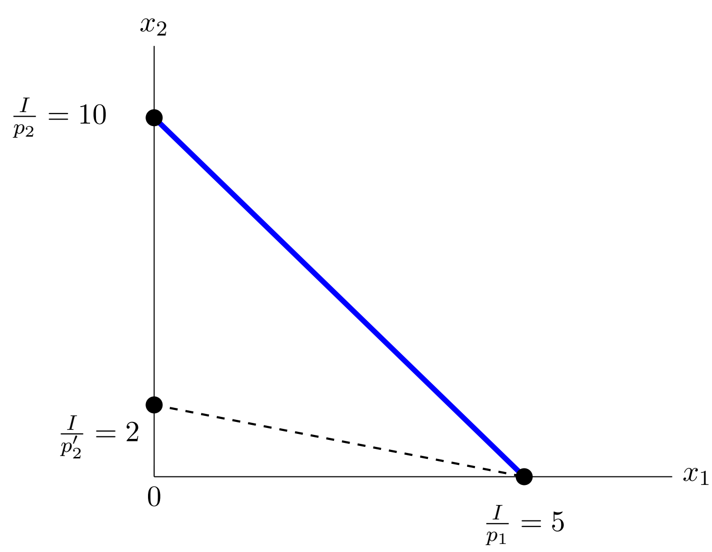
3.1 Examples

Example four: A consumer with an income of 20 chooses between two goods: x_1 and x_2. The corresponding prices are $p_1 = 4$ and $p_2 = 5$. Let's say that p_1 is reduced by 50% and falls to 2. Draw the new constraint.



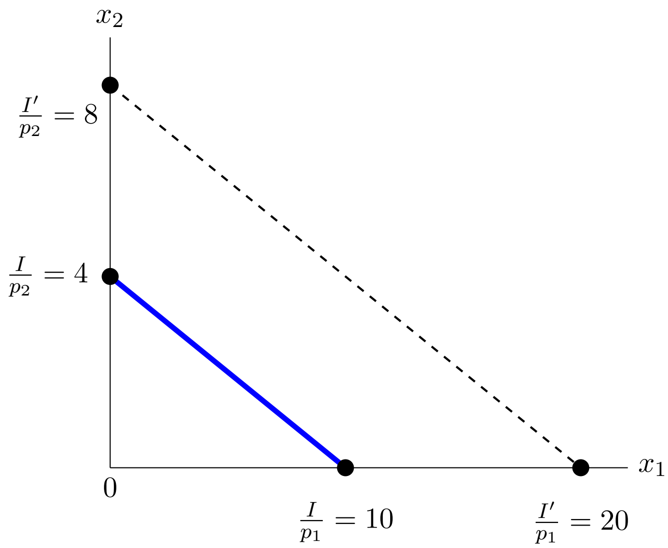
The slope in this case changes from $\frac{4}{20}$ to $\frac{2}{20}$, thereby making the constraint flatter. A flatter constraint means that we can afford more goods. Convince yourself of this. Do you see how this flatter, new constraint leads to more bundles being affordable? If not, email me.

Example five: A consumer with an income of 50 chooses between two goods: x_1 and x_2. The corresponding prices are $p_1 = 10$ and $p_2 = 5$. Let's say that p_2 is increased to 25. Draw the new constraint.



The slope in this case changes from $\frac{10}{5}$ to $\frac{10}{25}$, thereby making the constraint much, much steeper. A steeper constraint means that we can afford fewer goods. Convince yourself of this. We are restricting the number of bundles a consumer can afford.

Example six: A consumer with an income of 40 chooses between two goods: x_1 and x_2. The corresponding prices are $p_1 = 4$ and $p_2 = 10$. Let's say that our consumer wins the lottery and doubles their income.. Draw the new constraint.



Recall what we said earlier: if the income changes, we have a parallel shift. The slope is not changing at all. Therefore, we expect the budget constraint to pivot outward. Since our income doubled, we see that the number of goods we can afford in either direction (x and y) doubles as well.

4 Harder Parameter Changes

4.1 Non-linear Prices

The above examples are the most common in this class, so make sure to familiarize yourself with them. This section really concerns a few odd cases that are – in my opinion – much more realistic and interesting. The principles of the budget constraint are the same; we just need to get a bit more creative when thinking about how changes in price levels at different quantities affect the budget constraint.

Let's start by saying that $p_1 = \$3$ for the first 10 units of x. After the consumer purchases 10 units, the price increases to \$4. Let's slowly think through what patterns emerge when we change quantity purchase. If the consumer buys 5 units of x, they will pay $5x\$3 = \$15$. If they buy 10 units of x, they pay 10 x \$3 = \$30. If they buy 11 units of x, they pay $10x\$3 + 1x\$4 = \$34$. If they buy 12 units of x, they pay $10x\$3 + 2x\$4 = \$38$. Notice the pattern that develops? *Only after the 10 units* does the consumer face new prices, and accordingly, a new slope. This change in slope gives us a kinked budget constraint. Let's solve this one through: set $p_2 = 10$ and $I = 100$.

If we had constant prices, our budget constraint would be simple. It would have an intercept at $\frac{100}{3} = 33.33$ on the x-axis and an intercept at $\frac{100}{10} = 10$ on the y-axis. The slope of this line would be $\frac{-3}{10} = -0.3$.

What about after the price switch? Well, we already know that prices change *after* the tenth good, meaning we need to account for the fact that the consumer has already purchased 10 units of x at

$p_1 = 3$. This also implies that they have spent 10 x \$3 = \$30 already. Therefore, we write the new budget constraint like so:

$$4(x_1 - 10) + 3 * 10 + 10x_2 = 100$$

$$4x_1 - 40 + 30 + 10x_2 = 100$$

$$4x_1 + 10x_2 = 110$$

In general, the formula to account for changes in price (where new price is denoted by p') is:

$$p'(x_1 - \bar{x}_1) + p_1 * x_1 + p_2 * x_2 = I \tag{3}$$

Where is the kink, though? We know that at some point, the slope must change. Well, we already know at which x_1 value the slope changes. To find the corresponding x_2 value, though, we need to plug in the value of x_1 at which the slope changes.

$$p_1x_1 + p_2x_2 = I$$

$$30 + 10x_2 = 100$$

$$10x_2 = 70$$

$$x_2 = 7$$

This means that we have a kink at the point $(10, 7)$. The initial slope is $-.3$, and the new slope is $\frac{4}{10} = -.4$. Likewise, from $4x_1 + 10x_2 = 110$, we know that we should have intercepts at $(\frac{110}{4}, \frac{110}{10})$.

The plot below shows the relevant budget constraint.